Stochastic Model of Automatic Traffic Service Controlled by Lattice Coulomb Interactions

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Abstract: We define a model for a system of automatic vehicles (cybercars) circulating on arbitrary planar closed road-networks. The demands for transportation arrive according to a Poisson flow. The diffusion of cybercars along the network is influenced by Coulomb forces, determined by the distribution of clients in the system. This model is implemented numerically on a had-hoc graphical interface, using Monte-Carlo methods. In this way the *the Coulomb forces algorithm*, used to optimize the service, seems to be quite efficient. Preliminary numerical results are presented for this model, together with a phase diagram.

Key-words: Vehicular transport, exclusion process, thermodynamic limit, phase transition.

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Modèle Stochastique de Traffic Automatique Contrôlé par des Forces de Coulomb sur Réseau

Résumé : Nous définissons un modèle stochastique de véhicules automatiques (cybercars), circulant sur un réseau routier fermé, répondant à des appels survenants aléatoirement sur le réseau. Le mouvement des cybercars est orienté par des interactions de type Coulomb, induites par la distribution des clients en attente sur le réseau. Ce modèle est implémenté numériquement à l'aide d'une interface graphique had-hoc, permettant de visualiser les simulations Monte-Carlo, de façon à démontrer l'efficacité de l'algorithme. Des résultats numériques préliminaires concernant ce modèle sont présentés. Un diagramme de phase est proposés.

Mots-clés : Transport de véhicules, processus d'exclusion, limite thermodynamique, transition de phase.

1 Introduction

The increasing level of traffic in urban environment, leading to an unyielding congestion of city centers, encourages to reconsider public transportation for the future. Technological developments in the domain of automatization allow to propose radically new ways of transportations system. In this respect the Cybercars project is a proposal for full automation individual but public transport; a new public transportation system based on automatic vehicles. A lot of effort are nowadays devoted to the development of prototypes of such cars, based on different detections systems. In the future, a large number of such vehicles should be gathered to form fleets, in order to be able to transport at the same time, on a specific road network, a lot of authorized peoples (users) to some arbitrary destinations within the network. The management of such fleet, which should be fully automatic, requires the development of algorithms and models, in order to optimize the service, to be able to predict waiting-times, and transportation times. The typical constraint on a management system would be to minimize the waiting-times and the transportation times of users, while keeping the rate of utilization of cybercars quite reasonable. Modelization should provide tools for computing waiting-times and the optimized number of cars necessary to insure a service of a given quality. In this report we propose both a stochastic model based on the random occurrence of demands, and an algorithm designed to optimize redistribution of automatic cars on the network and able to avoid congestion for high-level demand situations. There is a wide range of models aimed at describing vehicular traffic flow (see [2] and reference herein). They range from microscopic or car following models, kinetic models based on Boltzmann type kinetic equations [6]and refinements, and macroscopic models using hydrodynamics equations like Navier-Stokes. More recently, methods coming from statistical physics have been used, and the asymmetric exclusion process [7] has been generalized for realistic purpose, leading to cellular automaton models [3, 4, 5]. In this report we propose to adapt this last approach to automatic transport networks.

2 The Model and the Algorithm

2.1 Elements of the model

Ingredients of the model are

(a) - the network: it is defined by a set of nodes with coordinates $\{(X_i, Y_i)\}, i = 1..N$ related with each others by oriented links with a given capacity C_{ij} and a unit length a independent of the link.

(b) - the users: they arrive at random on each node i of the network, according to some rate ϵ_i, and ask for a vehicle.

(c) - the cars: either occupied or unoccupied, they move on the links. On each link (ij) the number of cars cannot exceed the capacity C_{ij}. A car leaves the link (ij) for some link (jk) at a rate v_{ij}/a which represents the mean velocity on the link. The new destination k is taken randomly between all the neighbors of node i (except j the provenance node), according to some time-dependant probability $p_{jk}(t)$, which we will define later. When a client is waiting at site j, a free cybercar circulating on a link (i,j) will take him in charge. The destination is chosen randomly according to some weight w_k specified at each node k.

(d) - *home locations*: they are specific nodes where cybercars can park, from which they start and at which they stop. The rate at which cars can be delivered on the network are defined at each *home location* by ϵ_i (i being the corresponding node).

2.2 The Lattice Kernel

We propose in this report a unified way for controlling the service and insuring an optimized repartition of the vehicles on the network. The algorithm is based on an interacting field between cars and users which appear randomly on the system. Cars, users, destinations and *home locations* influence the system through the charge we associate to them. Once the charge are located on the network, the corresponding distributions $\rho_i(t)$ are convoluted to some kernel K_{ij}, which is specific to the network, and resulting fields $V(i,t)$ are obtained. In turn, these fields orientate the probabilities which determine the motions of the cars, and the rates at which cars are delivered or removed from the network. Before coming to this, let us look more specifically on the kernel K_{ij} which is the corner-stone of the present algorithm. By definition we let

$$K_{ij} = (\Delta + m^2)^{-1}_{ij}, \qquad 1 \le i \le N, \quad 1 \le j \le N.$$

Δ is the Laplace operator defined on the lattice. Namely we have

$$\Delta_{ij} = \sum_{k \in <i>} (\delta_{jk} - \delta_{ij}),$$

where $< i >$ are the nearest neighbors of i. This operator has no inverse, since the sum of its columns vanishes. Adding a small "mass" m^2 allow to regularize the Kernel K_{ij}. The correlation length $\xi = m^{-1}$ gives the range below which two nodes are correlated, and above which they are not correlated by the kernel K. This parameter will be referred as the range of the interaction and will be generally tuned such that it is greater than the size of the network.

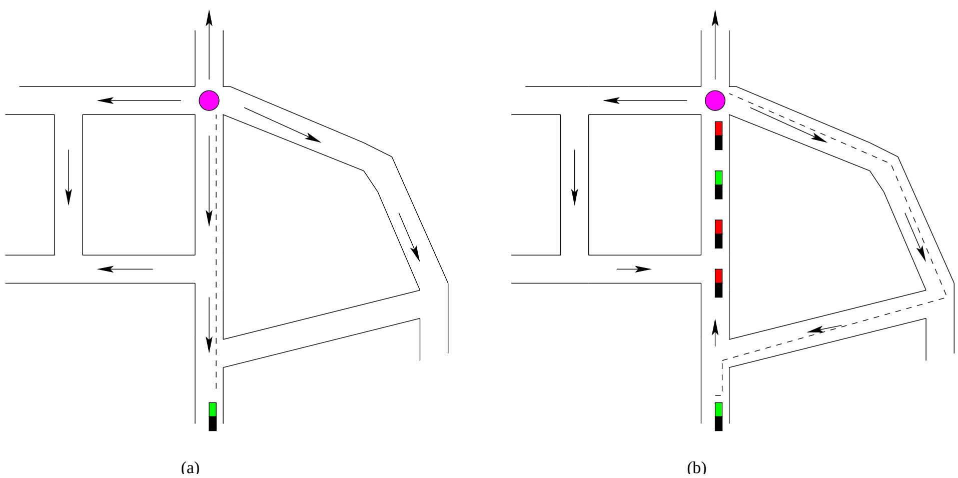


Fig. 1: Gradient of the field radiated by waiting customers (magenta stain) indicated by black arrows. (a) Direct deterministic path (dotted line) of the free car (green) obtained by going back up the field. (b) Detour due to screening by other cars.

2.3 Charges, fields and transitions probabilities

In our model, a charge q_1 is formally carried by the users and a charge q_2 by the cybercars, *home location* have a charge q_3 and destinations are affected with a charge

q_4. The field V which influence the behavior of the free cybercars is defined as follows

$$V(i) = \sum_{j\neq i}^{N} K_{ij}(q_1 n_1(j) + q_2 n_2(j) + q_3 n_3(j)), \tag{2.1}$$

with

$$\begin{cases} n_1(i) \stackrel{\text{def}}{=} \text{Number of users at site i} \\ n_2(i) \stackrel{\text{def}}{=} \text{Number of incomming cars to site i} \\ n_3(i) \stackrel{\text{def}}{=} \text{Number of free car park at a home location site i} \end{cases}$$

By convention, q_1 are going to be negative charges as well as q_3 and q_4, while q_2 will be positive. Once this field is defined the transitions probabilities are computed by means of Boltzmann weights. First of all, cars are delivered on the Network from a *home location* situated at site i with the rate

$$\epsilon_i(t) = \epsilon_i^0 \frac{\exp(-\beta V_i(t))}{1+\exp(-\beta V_i(t))},$$

where β is a so-called inverse temperature parameter, convenient to fix the amount of randomness we impose to the fleet and the reactivity to the demand. The rate $\epsilon_i(t)$ is designed in such a manner that the demand is evaluated from the sign of $V_i(t)$. Schematically, if $V_i > 0$ the cars are in excess, and in the contrary if $V_i < 0$, there are a lot of waiting users on the network compared to the amount of circulating cars; in that case $\epsilon_i(t)$ approaches its maximal value ϵ_i^0. At zero Temperature ($\beta \to \infty$) the fleet is fully reactive, either $V_i(t) > 0$, hence no vehicle can leave the *home location*, either $V_i(t) < 0$ then the rate is saturated to the value ϵ_i^0. Concerning the way cars are oriented through the network, we have to distinguish between free and occupied (i.e. carrying a user) cybercars. For an unoccupied car, the direction is drawn each time a cross-road is reached. A vehicle circulating on the link (ij) will chose the destination k pertaining to the set of nearest neighbors of j, with probability

$$p_{ij}(k) = \begin{cases} 0 & \text{if } k = i \\ \frac{1}{Z_{ij}} \exp(-\beta V_k(t)) & \text{otherwise} \end{cases} \tag{2.2}$$

with the normalization

$$Z_{ij} = \sum_{k\in<j>, k\neq i} \exp(-\beta V_k(t)).$$

We read from the expression of $V_k(t)$, that the first term will be responsible for the cybercars to be able to find their way towards waiting users; the second term will give to the cars the ability to avoid traffic jams; and the third term will insure a correct redistribution of cybercars among the *home location* by attracting the cars towards empty park sites. If the car is occupied, it is labeled with a certain destination d, which is remembered until this destination is reached. But again, unless $\beta \to \infty$, the motion of the cybercar is not deterministic; at cross-road j, coming from node i, the next node k is drawn according to (2.2), but with the field replaced by $q_4 K_{kd}$. Only the charge q_4 placed at the destination address does influence the route of the vehicle. When $\beta \to \infty$ the shortest distance route will be chosen. The drawback of this deterministic limit, as we will see, occurs for heavy traffic situations, where saturated links can cause bottlenecks to occur.

3 Numerical observations

3.1 The simulator

The numerical approach is based on a graphical interface, which allows to design and configure road networks, and to perform via Monte-Carlo simulations, some numerical experiments, by varying the parameter described preceedingly. These simulations are destinated to test the algorithm, by giving some quantitative observations measuring the efficiency of the system. Moreover it will allow to discrimate and classify different regimes, and to determine the relevant parameters driving the system in these various regimes, and responsible of the phases transitions which separates these regimes. Several parameters have to be tuned, which in the permanent regime, determine the state of the system, with the help of the following quantities

- the global rate of new demands,
- the total rate of service,
- the rate of car use,
- the effective speed of cars.

The global rate of new demands is simply the summations of all individuals rates of client arrivals at each node. Eventually these rates have a non-uniform distribution over the network. In the stationary regime this will be balance by the total rate of service which is limitate by the following factors: the available number of cars

Parameters	global(1) or local(2) Definition	Typical value
N	Number of nodes	1000
a	lenght of elementary links(1)	10m
$\alpha = \frac{v}{a}$	displacement rates of cars (1)	$0.5s^{-1}$
λ_i	Apparition rate of new clients (2)	$0.001s^{-1}$
ϵ_i	departure rate of cars (2)	$0.1s^{-1}$
$\xi = m^{-1}$	range of the interaction (1)	100 links
q_1, q_2 q_3, q_4	assigned charges to users, cars, destinations and free park sites (1)	1,1 10,0.1
$\beta = \frac{1}{kT}$	deterministic rate or "inverse temperature" (1)	2

able to circulate at the same time on the network, the *round-trip time*, i.e the time between the end of to successive courses. This quantity depends on the *reference speed* of cars, that is the speed at which a car can circulate in absence of others, and the traffic congestion. The finite capacity of links, which is a parameter in our model, in conjunction to a high level of demands, can cause queuing formations and traffic jams, and henceforth, a lowering of the speed of cars. In turn, this decreased effective speed, results in an increase of the *round-trip time*, and a reduction of the service rate. Reasons for non-ergodicity come from a local or a global impossibility of the service rate to balance the arrival intensity.

The performance of the system is described by the following quantities

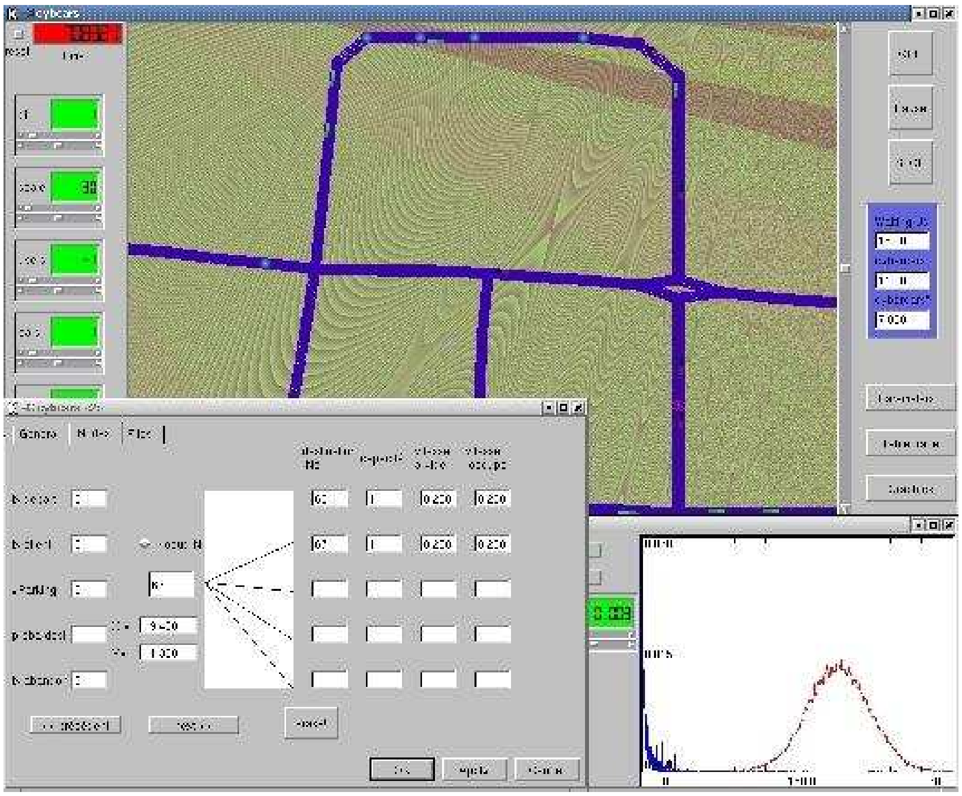


Fig. 2: Platform for numerical experiments. Green cars are free, red ones are occupied, red spots indicate waiting users and blue spots are destinations nodes for red cars.

- the waiting times of customers,
- the total service, i.e. the number of transported customers say, per hours,
- the rate of use of the cybercars, that is the number of circulating cars divided by the number of waiting customers.

3.2 Heuristic statements

The parameters and performance indicators defined previously are interplayed, and we will propose some heuristic relations among them, which we are able to observe numerically. These relations will be valid for homogeneous systems. Let us call λ the total rate of demands. For a lattice having N nodes we have

$$\lambda \stackrel{\text{def}}{=} \sum_{i=1}^{N} \lambda_i$$

with λ_i the rate of clients arrivals at each individual node i. In the ergodic regime, this global rate has to be exactly balanced by the service rate, which depends on the number of available cars, and on the traffic conditions. Let us denote by ρ,

$$\rho \stackrel{\text{def}}{=} \frac{\sum_{i=1}^{N} n_2(i)}{\sum_{i,j} C_{i,j}}$$

the density of cars circulating on the road network, with respect to the total capacity of the network. In the stationary regime we expect a relation of the form

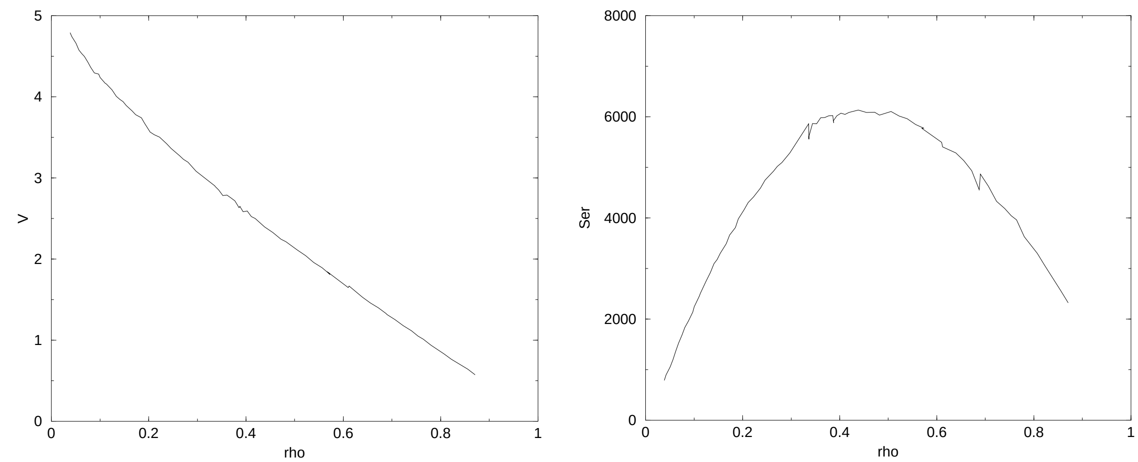
$$\rho = f(\frac{\lambda}{\lambda_m})$$

with f an increasing function of $x = \frac{\lambda}{\lambda_m}$, with a linear behavior at low rate, and a vertical slope at $x = 1$, expressing the maximum accessible rate λ_m. Above this value, the demand cannot be fulfilled, and the system is transient. Stated differently, expressing the total service rate as a function of ρ, this rate reaches its maximum for a given value ρ_M above which the limiting capacity of the network induces an increasing time service, resulting in a reduction of the service rate. This is a consequence of a structure relationship between the mean speed v of cars and the density on the network, referred as the *Fundamental diagram* [2]

$$v(\rho) = v_0\ g(\frac{\rho}{\rho_m})$$

where g is a positive decreasing function of $x \in [0, 1]$, vanishing at $x = 1$ and with $g(0) = 1$ Considering now the service rate, this has to be locally proportional to the traffic flow. Hence, still assuming a homogeneous system we have

$$\mu(\rho) = h\big(\rho v(\rho)\big),$$



(a) (b)

Fig. 3: Fundamental diagram expressing the behavior of the system when varying the density of cars on a $(6, 3)$ square lattice. (a) Velocity $(ms^{-1}$ as a function of ρ. (b) Service-rate μ $(\#clients/hours)$ as a function of ρ

with μ the global service rate, a monotonic increasing function h of the traffic flow vanishing when $\rho v = 0$. Therefore, considering μ as a function of ρ, the global rate vanishes for both extreme values $\rho = 0$ and $\rho = \rho_m \leq 1$, which are separated by a value ρ_M for which μ reaches its maximum value μ_M. This maximum value separates two different regimes of the system, since a same rate $\mu < \mu_M$ of requests can be fulfilled either by a low density or a high density distribution of cars (see figure 3.b and 5). Hence ρ_M is a transition point which separates a congested from a fluid regime to which the system adjusts itself for a given global rate λ of demands and a given size ρ_t of the fleet. The system can be in either phases, or can chose stochastically between the two. These various behavior depend on initial conditions and on the parameters

$$\gamma \stackrel{\text{def}}{=} \frac{q_1}{q_2}$$

and β. These two parameters do influence the system in the following way: First of all γ influences the number of cars which are delivered into the system from the *home-locations*, since in the stationary regime, the field V is likely to favour the neutrality of the system. Concerning β, depending on its value, it will render the

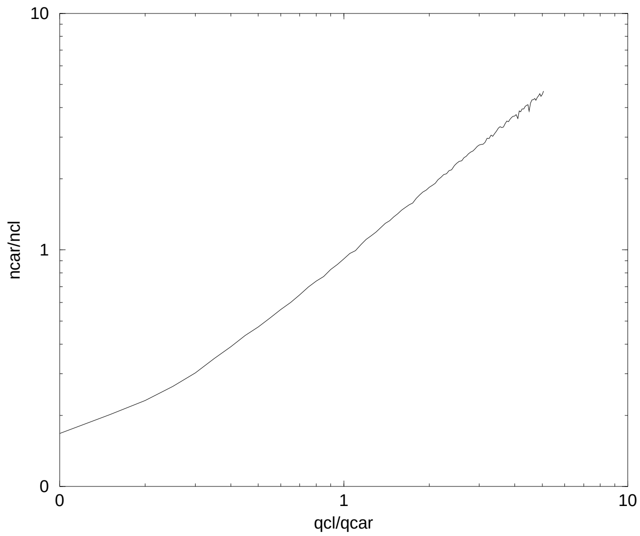


Fig. 4: Ratio of the number of clients with respect to the number of cars as a function of $\frac{q_1}{q_2}$ at low density on a $(6,3)$ square lattice (216 nodes).

control of the system more or less tight, and will in some way influence the amount of fluctuations of the system. Eventually a small β will cause sufficient fluctuations such that the system can oscillate between the low density and the high density regime.

3.3 Examples

3.3.1 Closed square lattice

Using the graphical interface, we can generate and configure networks with an arbitrary geometry. The most simple ones are square lattices. We have been able to test the redistribution algorithm in real time on such lattices, with up to 1000 nodes and several hundreds of cars circulating. Denote by (n,p) such a lattice with n the number of principal nodes (cross-roads) per side and p the number of intermediate nodes between cross-roads. The total number of nodes is $N = n[(n-1)(2p+1)+1]$ and the number of oriented links is $L = 4n(n-1)(p+1)$. The figures 3 and 4 correspond to preliminary studies on such a $(6,3)$ square lattice. Figures 3 indicate

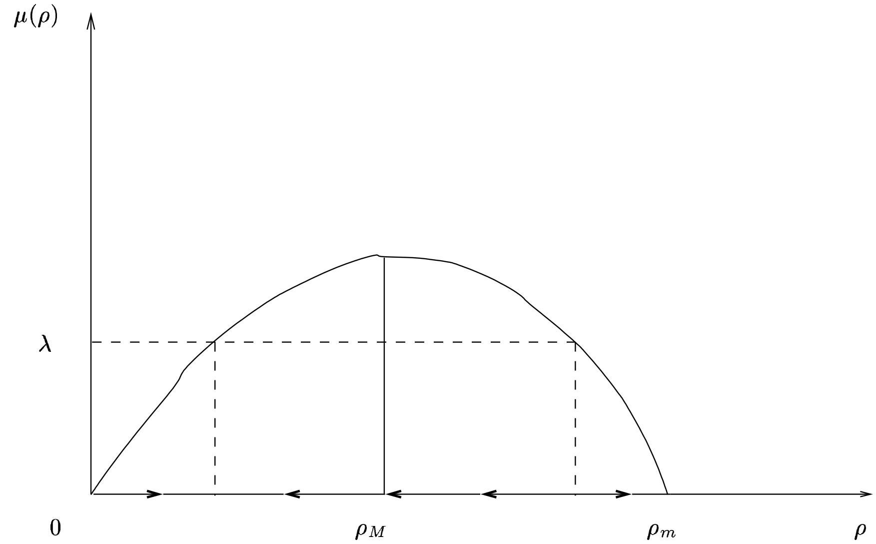


Fig. 5: Schematic phase diagram of the system. The system is ergodic below the curve, $\lambda < \mu(\rho_t)$ if ρ_t is the total fleet density. Arrows indicate the stability of the fixed points for a given λ

a linear relationship between velocity and car density at small β

$$v(\rho) = v_0(1 - \frac{\rho}{\rho_m}),$$

and a different behavior when β is increased. Accordingly $\rho_m$is actually a decreasing function of β with $\rho_m(0) = 1$. This shows that bottlenecks are more likely to swarm when increasing the deterministic parameter β. From figure 4 we see that at low density the ratio γ of the charges associated to clients and cars respectively, essentially determine the ratio $\frac{\#cars}{\#clients}$. This works like a neutrality principle in a distribution of charges.

3.3.2 The Inria network

We have tested the model on a network representing the part of the Inria site as it should be experimented in a near future (see figure 6). The network is represented by 69 nodes (cross-road and intermediate nodes) separated by links which represent a reference scale of 20 meters. One *home-location* supposed to have a capacity for

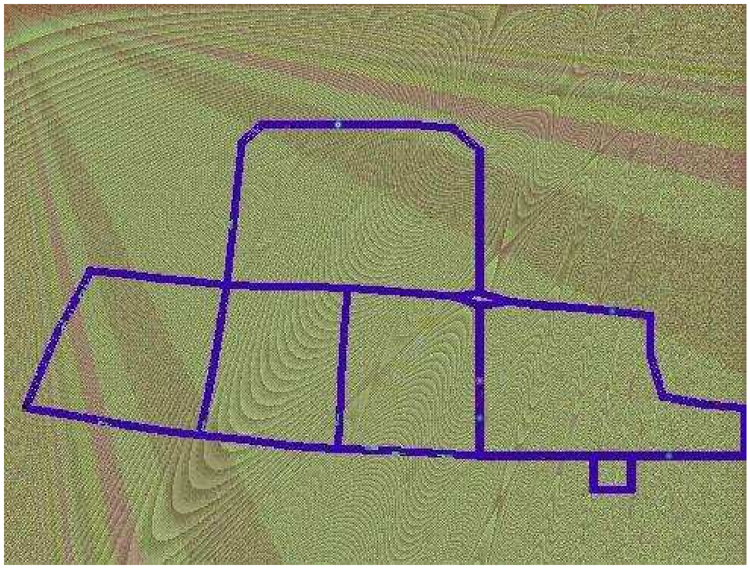


Fig. 6: Inria network with 69 nodes. Green cars are free looking for clients (red spots), red ones carry a users with a given destination (blue dots)

parking 30 cars is placed at the left end extremity of the network. Typically, a demand of the order of 300 hundred people per hours is successfully handled (the system is ergodic) by an average of 10 to 20 cars, circulating at a speed of $4ms^{-1}$, with waiting time not exceeding 1 to 3 minutes, depending on the ratio $r = \frac{\#q_2}{\#q_1}$. Different conditions for the repartition of the calls and the destinations have been tested. We have observed two different behavior of the system, characterized by the distribution of waiting-times (see figure 7). This is illustrated by two canonical situations;

- The demand is uniformly distributed over the network, at each node there is a rate of users apparitions (typically $0.002s^{-1}$ giving a global rate of $0.138s^{-1}$ calls)
- The demand is concentrated on one particular node (people going out of the canteen for example with a typical rate of $0.1s^{-1}$)

In the first case, waiting-times are distributed with some exponential law, instead, in the second case we observe a nodal distribution (see figure 7). These observations are not sensible on the way we draw the destinations i.e. whether we attribute a uniform heterogeneous distribution of destination weights on the networks. In the second case all cars are attracted to one node, where a queue of clients appear. The distribution is then centered around a value corresponding to the time users have to wait for their turn. When we combine this two types of distribution calls, the waiting-time distributions essentially superpose. Therefore, the presence of a nodal component in the waiting-times distribution is an indication that a queue is formed somewhere on the network.

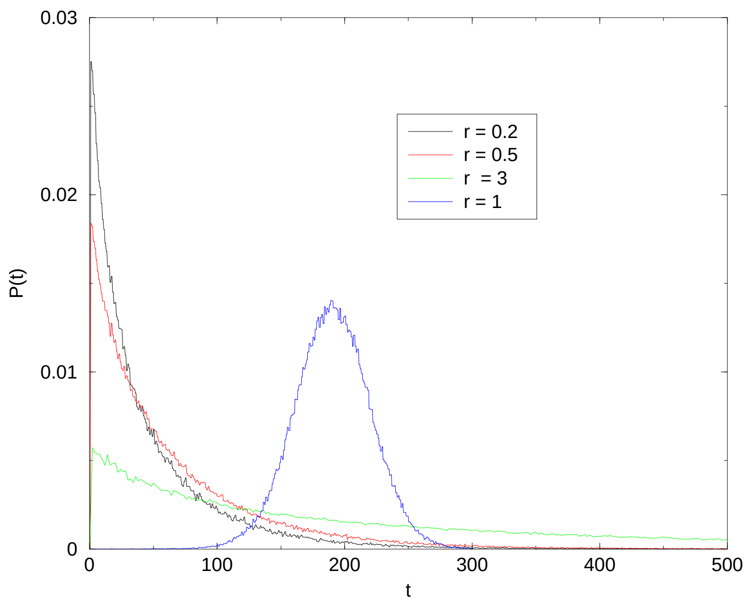


Fig. 7: Different distributions of waiting time accumulated for different values of the ratio $r = \frac{\#q_2}{\#q_1}$. First three curves correspond to a uniform demand, instead of a localized distribution of calls corresponding to the last curve.

4 Conclusion

We have presented here an algorithm for automatic cars service intended to respond to stochastic demands in a self-organized controlled way. The advantage of our algorithm is to avoid an individual treatment of each demand, and to replace it by a global regulation system represented by the interacting field V. Monte-Carlo simulations show effectiveness of the algorithm on arbitrary road-networks with up to 1000 nodes, with the ability to transport thousands of clients per hours with a few hundreds of cars with exponentially distributed waiting times, when tuning correctly a few control parameters. Preliminary numerical studies indicate a transition between a fluid and a congested phase when the fleet car density ρ_f exceeds a critical density ρ_c given by

$$\lambda = \mu(\rho_c),$$

at a given global request rate λ. This should be obtained by reformulating the model in terms of an effective queuing network model with asymptotic technics [8].

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